

Integrable geodesic flows on 2-torus: formal solutions and variational principle

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Abstract

In this paper we study quasi-linear system of partial differential equations which describes the existence of the polynomial in momenta first integral of the integrable geodesic flow on 2-torus. We proved in [3] that this is a semi-Hamiltonian system and we show here that the metric associated with the system is a metric of Egorov type. We use this fact in order to prove that in the case of integrals of degree three and four the system is in fact equivalent to a single remarkable equation of order 3 and 4 respectively. Remarkably the equation for the case of degree four has variational meaning: it is Euler-Lagrange equation of a variational principle. Next we prove that this equation for $n = 4$ has formal double periodic solutions as a series in a small parameter.

1 Introduction

In this paper we study integrable geodesic flows on two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$, where $\Gamma \subset \mathbb{Z}^2$ is a lattice. Let $ds^2 = \sum_{i,j=1}^2 g_{ij}(q) dq^i dq^j$ be a Riemannian metric on 2-torus. The geodesic flow of the metric is called integrable if the Hamiltonian system

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}^i = -\frac{\partial H}{\partial q^i}, \quad H = \frac{1}{2} \sum_{i,j=1}^2 g^{ij}(q) p_i p_j,$$

has a first integral $F(q, p) : T^*\mathbb{T}^2 \rightarrow \mathbb{R}$, i.e.

$$\dot{F} = \{F, H\} = \left(\frac{\partial H}{\partial q^1} \frac{\partial F}{\partial p_1} - \frac{\partial H}{\partial p_1} \frac{\partial F}{\partial q^1} \right) + \left(\frac{\partial H}{\partial q^2} \frac{\partial F}{\partial p_2} - \frac{\partial H}{\partial p_2} \frac{\partial F}{\partial q^2} \right) = 0,$$

such that almost everywhere F is independent with H . There are two kind of Riemannian metrics having polynomial integrals for the geodesic flow. If the metric has the form

$$ds^2 = \Lambda(\alpha x + \beta y)(dx^2 + dy^2) \text{ or } ds^2 = (\Lambda_1(\alpha_1 x + \beta_1 y) + \Lambda_2(\alpha_2 x + \beta_2 y))(dx^2 + dy^2),$$

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then there exist polynomial integrals of degree one or degree two. The existence of Riemannian metrics with non-reducible polynomial integrals of higher degree is a difficult open problem.

Let us recall some results on integrable geodesic flows on \mathbb{T}^2 . If the geodesic flow is integrable, then on the torus there are global semi-geodesic coordinates (t, x) (see [3]), i.e.

$$ds^2 = g^2(t, x)dt^2 + dx^2, \quad H = \frac{1}{2} \left(\frac{p_1^2}{g^2} + p_2^2 \right).$$

The polynomial first integral has the form

$$F = \frac{a_0}{g^n} p_1^n + \frac{a_1}{g^{n-1}} p_1^{n-1} + \dots + \frac{a_{n-2}}{g} p_1^2 p_2^{n-2} + p_1 p_2^{n-1} + p_2^n, \quad a_s = a_s(t, x).$$

The condition $\{F, H\} = 0$ is equivalent to the quasi-linear system of partial differential equations

$$U_t + A(U)U_x = 0, \tag{1}$$

where $U^\top = (a_0, \dots, a_{n-1})$, $a_{n-1} = g$,

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

System (1) has remarkable properties. It can be written in the form of conservation laws, i.e. there is a regular change of variables

$$(a_0, \dots, a_{n-1}) \mapsto (G_1(a), \dots, G_n(a))$$

such that for some $F_1(a), \dots, F_n(a)$ the conservation laws hold true

$$(G_i(a))_t + (F_i(a))_x = 0, \quad i = 1, \dots, n.$$

Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \dots, \lambda_n$ of A are real and pairwise distinct there exist change of variable

$$(a_0, \dots, a_{n-1}) \mapsto (r_1(a), \dots, r_n(a))$$

such that the system can be written in Riemann invariants.

$$(r_i)_t + \lambda_i(r)(r_i)_x = 0, \quad i = 1, \dots, n.$$

Such systems are called semi-Hamiltonian and generalized hodograph method applies [13]. Semi-Hamiltonian systems and in particular systems of hydrodynamic type are very important in mathematical physics (see for example [6], [7], [12]).

It is not clear to us at present if and how the generalized hodograph method can be used to prove non-existence of smooth solutions for a semi-Hamiltonian system. Using the original idea by P. Lax based on analysis along characteristics we proved for the cases $n = 3, 4$ that in the elliptic domain (where matrix A has two complex-conjugated eigenvalues) the behavior of solutions can be analyzed: the integrals of degree three and four are reduced to integrals of degree one or two [4]. Thus nontrivial integrals of degree 3, 4 may exist only in the hyperbolic region of the quasi-linear system (1).

In this paper we study a quasi-linear system (2) which corresponds to the choice of conformal coordinates (x, y) for Riemannian metric $ds^2 = \Lambda(dx^2 + dy^2)$. We assume that the geodesic flow has a polynomial in momenta integral

$$F = a_0(x, y)p_1^n + a_1(x, y)p_1^{n-1}p_2 + \cdots + a_n(x, y)p_2^n.$$

Kozlov and Denisova [8] proved that if Λ is trigonometric polynomial then the geodesic flow has no irreducible polynomial integrals of degree higher than two (see also [9]). By Kolokoltsov's [10] theorem

$$a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots,$$

where c_1, c_2 are some constants. Then the condition $\{H, F\} = 0$, where $H = \frac{p_1^2 + p_2^2}{2\Lambda}$ is equivalent to the system of quasi-linear equations

$$A(U)U_x + B(U)U_y = 0, \quad U = (a_0, a_1, \dots, a_{n-2}, \Lambda) \quad (2)$$

(see (11) below to specify $A(U), B(U)$ explicitly). This system also can be written in the form of conservation laws and moreover, in the hyperbolic region it admits n Riemann invariants, so the system is semi-Hamiltonian (see [2]).

Let us remind that for semi-Hamiltonian systems the following relations on eigenvalues hold

$$\partial_{r_j} \frac{\partial_{r_i} \lambda_k}{\lambda_i - \lambda_k} = \partial_{r_i} \frac{\partial_{r_j} \lambda_k}{\lambda_j - \lambda_k}, \quad i \neq j \neq k \neq i.$$

These relations mean that there exists a diagonal metric on the space of field variables

$$ds^2 = H_1^2(r)dr_1^2 + \cdots + H_n^2(r)dr_n^2 \quad (3)$$

with Christoffel symbols satisfying the identities

$$\Gamma_{ki}^k = \frac{\partial_{r_i} \lambda_k}{\lambda_i - \lambda_k}, \quad i \neq k.$$

Let us formulate now our main results. In Theorems 1–7 we assume that $c_1 = 0$. This can be achieved by a rotation in the plane x, y .

Theorem 1 *The metric (3) associated with the semi-Hamiltonian system (2) is a metric of Egorov type, i.e. the rotation coefficients*

$$\beta_{ij} = \frac{\partial_{r_i} H_j}{H_i}, \quad i \neq j$$

are symmetric $\beta_{ij} = \beta_{ji}$, or equivalently there is a function $A(r)$ such that

$$\partial_{r_i} A(r) = H_i^2(r).$$

In fact it follows from theorem of Pavlov and Tsarev [11] that in order to prove Theorem 1 one needs to find two conservation laws of a special form. In the next theorem we state the existence of these conservation laws for (2).

Theorem 2 *The system (2) has two conservation laws of the form*

$$P(U)_x + Q(U)_y = 0, \quad Q(U)_x + R(U)_y = 0. \quad (4)$$

Functions $P(U), Q(U), R(U)$ are found explicitly in Lemma 1 (see below). Remarkably, for $n = 3, 4$ Theorem 2 allows us to reduce system (2) to a single equation.

Theorem 3 *Let $n = 3$, and $\lambda(x, y)$ be a solution periodic with respect to the lattice Γ of the equation*

$$\Delta\lambda = \frac{3c_2}{2}\Lambda - 2a_{11} - 2a_{22}. \quad (5)$$

Then the function λ satisfies the equation

$$\begin{aligned} &2\lambda_{xx}\lambda_{xxy} + \lambda_{yy}(\lambda_{xxy} - \lambda_{yyy}) + \lambda_{xy}(\lambda_{xxx} + \lambda_{xyy}) + 4a_{11}\lambda_{xxy} + \\ &2a_{12}(\lambda_{xxx} + \lambda_{xyy}) + 2a_{22}(\lambda_{xxy} - \lambda_{yyy}) = 0, \end{aligned} \quad (6)$$

where a_{11}, a_{12}, a_{22} are some constants defined by the metric and the integral.

Theorem 4 *Let $n = 4$, and $\lambda(x, y)$ be a solution periodic with respect to the lattice Γ of the equation*

$$\Delta\lambda = 2c_2\Lambda - 2a_{11} - 2a_{22}.$$

Then the function λ satisfies the equation

$$\begin{aligned} &\lambda_{xy}(\lambda_{yyyy} - \lambda_{xxxx}) + 3(\lambda_{yyy}\lambda_{xyy} - \lambda_{xxy}\lambda_{xxx}) + \\ &+ 2(\lambda_{yy}\lambda_{xyyy} - \lambda_{xx}\lambda_{xxxy}) + 4a_{22}\lambda_{xyyy} - 4a_{11}\lambda_{xxxy} + 2a_{12}(\lambda_{yyyy} - \lambda_{xxxx}) = 0, \end{aligned} \quad (7)$$

where a_{11}, a_{12}, a_{22} are some constants defined by the metric and the integral.

Our next theorem states that the equations (6) and (7) are in fact equivalent to the system (2).

Theorem 5 *Let λ be a solution of the equation (6) or (7) respectively periodic with respect to the lattice Γ which satisfies the condition $\Delta\lambda + 2(a_{11} + a_{22}) > 0$. Then the corresponding solution of the system (2) is periodic also.*

Proof of this theorem is very simple for the case $n = 3$ but requires certain topological argument for the case $n = 4$. This argument is given below in Section 4.

Remarkably equation (7) admits the following variational interpretation.

Theorem 6 *Equation (7) coincides with the Euler-Lagrange equation of the functional*

$$\mathcal{L}(\lambda) = \int \frac{1}{2} (4\lambda_{xy}(a_{22}\lambda_{yy} - a_{11}\lambda_{xx}) + 2a_{12}(\lambda_{yy}^2 - \lambda_{xx}^2) + \lambda_{xy}(\lambda_{yy}^2 - \lambda_{xx}^2)) dx dy$$

Let us remark that the functional becomes especially simple

$$\mathcal{L}(\lambda) = \int \frac{1}{2} \lambda_{xy}(\lambda_{yy} - \lambda_{xx}) \left(\frac{1}{\varepsilon} + \Delta \lambda \right) dx dy$$

for the choice of constants a_{ij} where $a_{11} = a_{22} = \frac{1}{4\varepsilon}$, $a_{12} = 0$. However, at the present moment we have no significant results on the critical points of this functional.

Let us consider now in more details the equation (7) for $a_{12} = 0$, $a_{11} = a_{22} = \frac{1}{4\varepsilon}$

$$\begin{aligned} \lambda_{xxxy} - \lambda_{xyyy} = \varepsilon(\lambda_{xy}(\lambda_{yyyy} - \lambda_{xxxx}) + 3(\lambda_{yyy}\lambda_{xyy} - \lambda_{xxy}\lambda_{xxx}) + \\ + 2(\lambda_{yy}\lambda_{xyyy} - \lambda_{xx}\lambda_{xxxy})). \end{aligned} \quad (8)$$

The conformal factor of the metric has the form

$$\Lambda = \frac{1}{c_2} \left(\frac{\Delta \lambda}{2} + \frac{1}{2\varepsilon} \right).$$

Let us look for a solution of (8) as a formal power series in ε :

$$\lambda(x, y) = \lambda_0(x, y) + \lambda_1(x, y)\varepsilon + \lambda_2(x, y)\varepsilon^2 + \dots, \quad (9)$$

where ε is a small parameter. Then from (8) we have a recursion formula.

$$(\lambda_k)_{xyyy} - (\lambda_k)_{xxxy} = \sum_{s=0}^{k-1} \langle \lambda_s, \lambda_{k-s-1} \rangle \quad (10)$$

where

$$\langle \lambda_p, \lambda_q \rangle = \lambda_{pxy}(\lambda_{qyyyy} - \lambda_{qxxxx}) + 3(\lambda_{pyyy}\lambda_{qxyy} - \lambda_{pxxx}\lambda_{qxxy}) + 2(\lambda_{pyy}\lambda_{qxyyy} - \lambda_{pxx}\lambda_{qxxxy}).$$

In the rest of this section we shall assume that $\Gamma \subset \mathbb{R}^2$ is the integer lattice \mathbb{Z}^2 . Given initial doubly periodic function λ_0 we wish to find all λ_k recursively by means of equation (10). It is easy to see that (10) has a periodic solution λ_k if the Fourier series of the right hand side does not have monomials of the form

$$e^{inx}, \quad e^{iny}, \quad e^{in(x+y)}, \quad e^{in(x-y)}.$$

Let us remark also that the periodic solution λ_k of (10) is defined up to addition to λ_k some function of the form

$$\tilde{\lambda}_k = f_1(x) + f_2(y) + f_3(x - y) + f_4(x + y).$$

Next theorem states that recursive process is well defined and thus gives formal periodic solution of (8) if the initial function λ_0 and the additions $\tilde{\lambda}_k$ on every stage k are chosen symmetric with respect to coordinate axes and diagonals, $x = 0, y = 0, x = y, x = -y$. It is an important open question if the convergence of the series can be achieved by a good choice of initial function λ_0 and the functions $\tilde{\lambda}_k$.

Theorem 7 *Let*

$$\lambda_0 = \sum_{n \in \mathbb{N}} \alpha_n (\cos(nx) + \cos(ny)) + \sum_{n \in \mathbb{N}} \beta_n (\cos(n(x - y)) + \cos(n(x + y))),$$

$$\tilde{\lambda}_k = \sum_{n \in \mathbb{N}} \alpha_n^k (\cos(nx) + \cos(ny)) + \sum_{n \in \mathbb{N}} \beta_n^k (\cos(n(x - y)) + \cos(n(x + y))),$$

then the recursion formula gives a well defined formal periodic solution (9) of (10).

We prove Theorem 7 in the last section.

Remark *It is an interesting fact that substituting into the equation (8)*

$$\lambda = f_1(x) + f_2(y) + f_3(x - y) + f_4(x + y)$$

one gets an equation on the functions f_1, \dots, f_4 which was studied recently in [1]. It was proved later in [5] that there are no new periodic solutions of this equation in this particular form.

2 Proof of Theorems 1 and 2

Let us assume that $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ is a metric on \mathbb{T}^2 and F is a first integral polynomial in momenta for the geodesic flow

$$F = a_0(x, y)p_1^n + a_1(x, y)p_1^{n-1}p_2 + \dots + a_n(x, y)p_2^n.$$

We have

$$\begin{aligned} 2\Lambda^2\{H, F\} = & 2(a_{0_x}p_1^n + a_{1_x}p_1^{n-1}p_2 + a_{2_x}p_1^{n-2}p_2^2 + \dots + a_{n_x}p_2^n)p_1\Lambda + \\ & +(na_0p_1^{n-1} + (n-1)a_1p_1^{n-2}p_2 + (n-2)a_2p_1^{n-3}p_2^2 + \dots + a_{n-1}p_2^{n-1})(p_1^2 + p_2^2)\Lambda_x + \\ & 2(a_{0_y}p_1^n + a_{1_y}p_1^{n-1}p_2 + a_{2_y}p_1^{n-2}p_2^2 + \dots + a_{n_y}p_2^n)p_2\Lambda + \\ & +(a_1p_1^{n-1} + 2a_2p_1^{n-2}p_2 + 3a_3p_1^{n-3}p_2^2 + \dots + na_np_2^{n-1})(p_1^2 + p_2^2)\Lambda_y. \end{aligned}$$

Let us denote by l_{n+1}, \dots, l_0 coefficients of the homogeneous polynomial $2\Lambda^2\{H, F\}$

$$2\Lambda^2\{H, F\} = l_{n+1}p_1^{n+1} + l_n p_1^n p_2 + \dots + l_0 p_2^{n+1},$$

$$l_{n+1} = 2a_{0_x}\Lambda + na_{0_x}\Lambda_x + a_1\Lambda_y,$$

$$l_n = 2a_{1_x}\Lambda + (n-1)a_1\Lambda_x + 2a_{0_y}\Lambda + 2a_2\Lambda_y,$$

$$l_{n-1} = 2a_{2_x}\Lambda + na_{0_x}\Lambda_x + (n-2)a_2\Lambda_x + 2a_{1_y}\Lambda + a_1\Lambda_y + 3a_3\Lambda_y,$$

$$l_{n-2} = 2a_{3_x}\Lambda + (n-1)a_1\Lambda_x + (n-3)a_3\Lambda_x + 2a_{2_y}\Lambda + 2a_2\Lambda_y + 4a_4\Lambda_y,$$

$$l_{n-3} = 2a_{4_x}\Lambda + (n-2)a_2\Lambda_x + (n-4)a_4\Lambda_x + 2a_{3_y}\Lambda + 3a_3\Lambda_y + 5a_5\Lambda_y,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$l_3 = 2a_{n-2_x}\Lambda + 4a_{n-4}\Lambda_x + 2a_{n-2}\Lambda_x + 2a_{n-3_y}\Lambda + (n-3)a_{n-3}\Lambda_y + (n-1)a_{n-1}\Lambda_y,$$

$$l_2 = 2a_{n-1_x}\Lambda + 3a_{n-3}\Lambda_x + a_{n-1}\Lambda_x + 2a_{n-2_y}\Lambda + (n-2)a_{n-2}\Lambda_y + na_n\Lambda_y,$$

$$l_1 = 2a_{n_x}\Lambda + 2a_{n-2}\Lambda_x + 2a_{n-1_y}\Lambda + (n-1)a_{n-1}\Lambda_y,$$

$$l_0 = a_{n-1}\Lambda_x + 2a_{n_y}\Lambda + na_n\Lambda_y.$$

Let us recall that by Kolokoltsov's theorem

$$a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \dots, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \dots$$

We have the following system of differential equations

$$l_{n+1} = \dots = l_0 = 0. \tag{11}$$

Lemma 1 *If n is even $n = 2k$, then the system (11) has the following two conservation laws*

$$\begin{aligned} & [(na_0 - (n-2)a_2 + (n-4)a_4 - \dots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\ & [(-(n-2)a_1 + (n-4)a_3 - \dots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n-1)c_1)\Lambda]_y = 0, \\ & [((n-2)a_1 - (n-4)a_3 + \dots + (-1)^k 2a_{n-3} + (-1)^{k+1}c_1)\Lambda]_x + \\ & [(na_0 - (n-2)a_2 + (n-4)a_4 - \dots + (-1)^{k+1}2a_{n-2} + (-1)^{k+1}nc_2)\Lambda]_y = 0. \end{aligned}$$

If n is odd $n = 2k + 1$ then the system (11) has the following two conservation laws

$$\begin{aligned} & [((n-1)a_0 - (n-3)a_2 + (n-5)a_4 - \dots + (-1)^{k+1}2a_{n-3} + (-1)^k c_1)\Lambda]_x + \\ & [(-(n-1)a_1 + (n-3)a_3 - (n-5)a_5 + \dots + (-1)^k 2a_{n-2} + (-1)^k nc_2)\Lambda]_y = 0, \\ & [((n-1)a_1 - (n-3)a_3 + \dots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\ & [((n-1)a_0 - (n-3)a_2 + \dots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n-1)c_1)\Lambda]_y = 0. \end{aligned}$$

Theorems 1 and 2 immediately follow from the Lemma 1. Indeed, if we put $c_1 = 0$ in Lemma 1 we get two conservation laws of the form (4), where for n even

$$P = (na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda,$$

$$Q = (-(n-2)a_1 + (n-4)a_3 - \cdots + (-1)^{k+1}2a_{n-3})\Lambda,$$

$$R = (-na_0 + (n-2)a_2 - (n-4)a_4 + \cdots + (-1)^k 2a_{n-2} + (-1)^k n c_2)\Lambda.$$

Analogously for n odd one has:

$$P = ((n-1)a_1 - (n-3)a_3 + \cdots + (-1)^{k+1}2a_{n-2})\Lambda,$$

$$Q = ((n-1)a_0 - (n-3)a_2 + \cdots + (-1)^{k+1}2a_{n-3})\Lambda,$$

$$R = (-(n-1)a_1 + (n-3)a_3 - (n-5)a_5 + \cdots + (-1)^k 2a_{n-2} + (-1)^k n c_2)\Lambda.$$

Proof of Lemma 1

Let us consider the case of even degree $n = 2k$. Then

$$a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^k a_1, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots + (-1)^{k+1} a_0.$$

Let us consider the following linear combination of equations (11)

$$\begin{aligned} & nl_{n+1} - (n-2)l_{n-1} + (n-4)l_{n-3} - \cdots + (-1)^{k+1}2l_3 = \\ & 2(na_{0_x} - (n-2)a_{2_x} + (n-4)a_{4_x} - \cdots + (-1)^{k+1}2a_{n-2_x})\Lambda + \\ & (n^2a_0 - (n-2)na_0 - (n-2)^2a_2 + (n-4)(n-2)a_2 + (n-4)^2a_4 - \cdots \\ & \quad + (-1)^{k+1}2 \cdot 4a_{n-4} + (-1)^{k+1}2^2a_{n-2})\Lambda_x + \\ & 2(-(n-2)a_{1_y} + (n-4)a_{3_y} - \cdots + (-1)^{k+1}2a_{n-3_y})\Lambda + \\ & (na_1 - (n-2)a_1 - (n-2)3a_3 + (n-4)3a_3 + (n-4)5a_5 - \cdots + (-1)^{k+1}2(n-3)a_{n-3} + \\ & \quad (-1)^{k+1}2(n-1)a_{n-1})\Lambda_y = \\ & 2(na_{0_x} - (n-2)a_{2_x} + (n-4)a_{4_x} - \cdots + (-1)^{k+1}2a_{n-2_x})\Lambda + \\ & 2(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda_x + \\ & 2(-(n-2)a_{1_y} + (n-4)a_{3_y} - \cdots + (-1)^{k+1}2a_{n-3_y})\Lambda + \\ & 2(a_1 - 3a_3 + 5a_5 - \cdots + (-1)^k(n-3)a_{n-3} + (-1)^{k+1}a_{n-1}(n-1))\Lambda_y = \\ & 2[(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\ & 2(-(n-2)a_{1_y} + (n-4)a_{3_y} - \cdots + (-1)^{k+1}2a_{n-3_y})\Lambda + \\ & 2(a_1 - 3a_3 + 5a_5 - \cdots + (-1)^k(n-3)a_{n-3} + (-1)^{k+1}(n-1)(c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^k a_1))\Lambda_y = \\ & 2[(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\ & 2(-(n-2)a_{1_y} + (n-4)a_{3_y} - \cdots + (-1)^{k+1}2a_{n-3_y})\Lambda + \end{aligned}$$

$$\begin{aligned}
& 2(-(n-2)a_1 + (n-4)a_3 - \cdots + (-1)^{k+1}2a_{n-3})\Lambda_y + 2(-1)^{k+1}(n-1)c_1\Lambda_y = \\
& 2[(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\
& 2[(-(n-2)a_1 + (n-4)a_3 - \cdots + (-1)^{k+1}2a_{n-3})\Lambda]_y + 2(-1)^{k+1}(n-1)c_1\Lambda_y = 0.
\end{aligned}$$

We have the first required conservation law. By similar calculations,

$$\begin{aligned}
& nl_n - (n-2)l_{n-2} + (n-4)l_{n-4} + \cdots + (-1)^{k+1}2l_2 = \\
& 2[((n-2)a_1 - (n-4)a_3 + \cdots + (-1)^k2a_{n-3} + (-1)^{k+1}c_1)\Lambda]_x + \\
& 2[(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2} + (-1)^{k+1}nc_2)\Lambda]_y = 0.
\end{aligned}$$

Let us consider the case of odd degree $n = 2k + 1$. Then

$$a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^{k+1}a_0, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots + (-1)^{k+1}a_1.$$

By direct calculation we get

$$\begin{aligned}
& (n+1)l_{n+1} - (n-1)l_{n-1} + (n-3)l_{n-3} - \cdots + (-1)^k2l_2 = \\
& 2[((n-1)a_0 - (n-3)a_2 + (n-5)a_4 - \cdots + (-1)^{k+1}2a_{n-3} + (-1)^k c_1)\Lambda]_x + \\
& 2[(-(n-1)a_1 + (n-3)a_3 - (n-5)a_5 + \cdots + (-1)^k2a_{n-2} + (-1)^k nc_2)\Lambda]_y = 0,
\end{aligned}$$

and

$$\begin{aligned}
& (n-1)l_n - (n-3)l_{n-2} + (n-5)l_{n-4} - \cdots + (-1)^{k+1}2l_3 = \\
& 2[((n-1)a_1 - (n-3)a_3 + \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \\
& 2[((n-1)a_0 - (n-3)a_2 + \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n-1)c_1)\Lambda]_y = 0.
\end{aligned}$$

Lemma 1 and Theorems 1,2 are proved.

3 Proof of Theorems 3 and 4

Let us consider the case $c_1 = 0, n = 3$ in Lemma 1. Then we have conservation laws (4) where

$$P = -a_1\Lambda, \quad Q = -a_0\Lambda, \quad R = \left(\frac{3}{2}c_2 + a_1\right)\Lambda. \quad (12)$$

From (4) it follows that

$$P = h_{yy}, \quad Q = -h_{xy}, \quad R = h_{xx}, \quad (13)$$

where $h(x, y)$ is some function. From (12) and (13) we obtain

$$a_0 = \frac{3c_2 h_{xy}}{2\Delta h}, \quad a_1 = -\frac{3c_2 h_{yy}}{2\Delta h}, \quad \Lambda = \frac{2\Delta h}{3c_2}. \quad (14)$$

For $n = 3$ the equation $l_4 = 0$ has the form

$$a_1\Lambda_y + 2a_{0x}\Lambda + 3a_0\Lambda_x = 0.$$

Let us substitute (14) in the last equation. We get

$$2h_{xx}h_{xxy} + h_{yy}(h_{xxy} - h_{yyy}) + h_{xy}(h_{xxx} + h_{xyy}) = 0.$$

Since a_0, a_1, Λ are periodic function we have

$$h = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$

where λ is a function periodic with respect to Γ . Then λ satisfies the equation (6). Theorem 3 is proved.

Let us consider the case $c_1 = 0, n = 4$ in Lemma 1. Then we have conservation laws (4) where

$$P = (2a_0 - a_2)\Lambda, \quad Q = -a_1\Lambda, \quad R = (2c_2 - 2a_0 + a_2)\Lambda. \quad (15)$$

We have

$$P = f_{yy}, \quad Q = -f_{xy}, \quad R = f_{xx}, \quad (16)$$

where $f(x, y)$ is some function. From (15), (16) we get

$$a_1 = 2c_2 \frac{f_{xy}}{\Delta f}, \quad a_2 = -2c_2 \frac{f_{yy}}{\Delta f} + 2a_0, \quad \Lambda = \frac{\Delta f}{2c_2}. \quad (17)$$

Using (17) from $l_5 = 0$ and $l_4 = 0$ we get

$$a_{0x} = -\frac{1}{(\Delta f)^2} (c_2 f_{yyy} f_{xy} + c_2 f_{xy} f_{xxy} + 2a_0 \Delta f (f_{xxx} + f_{xyy})), \quad (18)$$

$$\begin{aligned} a_{0y} &= \frac{1}{(\Delta f)^2} (2f_{yy}((c_2 - a_0)f_{yyy} - a_0 f_{xxy}) \\ &\quad - 2f_{xx}(c_2 f_{xxy} + a_0(f_{yyy} + f_{xxy})) - c_2 f_{xy}(f_{xyy} + f_{xxx})). \end{aligned} \quad (19)$$

We differentiate (18) with respect to y , (19) — with respect to x and take a difference between the results, and after that we substitute into the result instead of a_{0x} and a_{0y} expressions (18) and (19). It gives an equation on f

$$f_{xy}(f_{yyyy} - f_{xxx}) + 3(f_{yyy}f_{xyy} - f_{xxx}f_{xxy}) + 2(f_{yy}f_{xyyy} - f_{xx}f_{xxy}) = 0.$$

Since P, Q and R are periodic functions, f can be written in the form

$$f = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$

where λ is a function periodic with respect to Γ . This yields (7). Theorem 4 is proved.

4 Proof of Theorems 5 and 6

Let us prove first Theorem 5. We start with the simple case of $n = 3$. Given a periodic solution λ of equation (6) satisfying $\Delta\lambda + 2(a_{11} + a_{22}) > 0$, then it follows from the explicit formulas (14) that the coefficients of the integral a_0, a_1, a_2, a_3 as well as the factor Λ are periodic functions.

In order to treat the case $n = 4$ we proceed with a topological argument as follows. Let λ be a periodic solution of the equation (7) satisfying $\Delta\lambda + 2(a_{11} + a_{22}) > 0$. Coefficients of the integral and the conformal factor Λ are determined by the function

$$f = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

using the formulas (17), (18), (19). Notice that by Kolokoltsov identities and (17) the functions Λ, a_1, a_3 are periodic with respect to the lattice. However, coefficients a_0 and also a_2, a_4 are not necessarily periodic. Coefficient a_0 is determined by the equations (18), (19) and it is convenient to rewrite them in the form

$$(a_0(\Delta f)^2)_x = -c_2 f_{xy}(\Delta f)_y := V \quad (20)$$

$$(a_0(\Delta f)^2)_y = c_2(f_{yy}^2 - f_{xx}^2)_y - c_2 f_{xy}(\Delta f)_x := W. \quad (21)$$

The functions $\Delta f, V, W$ are periodic and we need to show that the periods of the 1-form $Vdx + Wdy$ on the torus are zeroes. We have from (20), (21)

$$a_0 = \frac{L(x, y)}{(\Delta f)^2} + \frac{P_0(x, y)}{(\Delta f)^2},$$

where L is a linear function and P_0 and Δf are periodic functions with respect to the lattice. Due to (17) we have for other coefficients an analogous form:

$$a_2 = \frac{2L(x, y)}{(\Delta f)^2} + \frac{P_2(x, y)}{(\Delta f)^2},$$

and since $a_4 = c_2 + a_2 - a_0$ we get also

$$a_4 = \frac{L(x, y)}{(\Delta f)^2} + \frac{P_4(x, y)}{(\Delta f)^2},$$

where P_2, P_4 are periodic functions. In addition, odd coefficients a_1, a_3 are periodic functions. Take now two periodic geodesics γ_1, γ_2 on the covering plane of the configuration torus representing two independent homotopy classes e_1, e_2 of the lattice. Denote by z the intersection point and by

$$z_1 = z + e_1, \quad z_2 = z + e_2$$

the translations of z . Since

$$F = a_0(x, y)p_1^4 + a_1(x, y)p_1^3p_2 + a_2(x, y)p_1^2p_2^2 + a_3(x, y)p_1p_2^3 + a_4(x, y)p_2^4,$$

is the first integral of the geodesic flow we have that the increment of F along the two geodesics γ_1, γ_2 must vanish. But on the other hand we compute:

$$\begin{aligned}\Delta F|_{\gamma_1} &= \frac{L(e_1)}{(\Delta f)^2} (p'_1)^4 + \frac{2L(e_1)}{(\Delta f)^2} (p'_1)^2 (p'_2)^2 + \frac{L(e_1)}{(\Delta f)^2} (p'_2)^4, \\ \Delta F|_{\gamma_2} &= \frac{L(e_2)}{(\Delta f)^2} (p''_1)^4 + \frac{2L(e_2)}{(\Delta f)^2} (p''_1)^2 (p''_2)^2 + \frac{L(e_2)}{(\Delta f)^2} (p''_2)^4,\end{aligned}$$

where we used the form of the coefficients a_i . Here we used (p'_1, p'_2) (respectively (p''_1, p''_2)) for the momenta variables corresponding to the tangent vector $\dot{\gamma}_1(z)$ (respectively $\dot{\gamma}_2(z)$) at the intersection point z . But the last two identities reduce to

$$\begin{aligned}\Delta F|_{\gamma_1} &= \frac{L(e_1)}{(\Delta f)^2} ((p'_1)^2 + (p'_2)^2)^2 = 0, \\ \Delta F|_{\gamma_2} &= \frac{L(e_2)}{(\Delta f)^2} ((p''_1)^2 + (p''_2)^2)^2 = 0.\end{aligned}$$

Thus

$$L(e_1) = L(e_2) = 0,$$

which means that the linear function L vanishes. This completes the proof of Theorem 5 for $n = 4$.

Let us finish this section establishing variational form of the equation (7). This becomes clear if one rewrites (7) in the following way:

$$\begin{aligned}4a_{22}\lambda_{xyyy} - 4a_{11}\lambda_{xxxy} + 2a_{12}(\lambda_{yyyy} - \lambda_{xxxx}) + \\ + (\lambda_{xy}\lambda_{yy})_{yy} - (\lambda_{xy}\lambda_{xx})_{xx} + \frac{1}{2}(\lambda_{yy}^2 - \lambda_{xx}^2)_{xy} = 0.\end{aligned}\tag{22}$$

It is easy to verify that last equation is indeed Euler-Lagrange equation of the functional of Theorem 6. This completes the proof.

5 Proof of Theorem 7

As in the proof of Theorem 6 let us rewrite equation (8) in the form

$$\begin{aligned}\lambda_{xxxy} - \lambda_{yyyx} = \\ = \varepsilon(\lambda_{xy}\lambda_{yy})_{yy} - (\lambda_{xy}\lambda_{xx})_{xx} + \frac{1}{2}(\lambda_{yy}^2 - \lambda_{xx}^2)_{xy}.\end{aligned}\tag{23}$$

Then the recursion step (10) looks as follows:

$$(\lambda_k)_{xxxy} - (\lambda_k)_{yyyx} = \sum_{p,q \geq 0, p+q=k-1}^{k-1} A_{pq} + B_{pq} + C_{pq},\tag{24}$$

where

$$A_{pq} = ((\lambda_p)_{xy}(\lambda_q)_{yy})_{yy},$$

$$B_{pq} = -((\lambda_p)_{xy}(\lambda_q)_{xx})_{xx},$$

$$C_{pq} = \frac{1}{2}((\lambda_p)_{yy}(\lambda_q)_{yy} - (\lambda_p)_{xx}(\lambda_q)_{xx})_{xy}.$$

We prove by induction the following claim. All λ_k have no Fourier monomials of the form

$$e^{inx}, \quad e^{iny}, \quad e^{in(x+y)}, \quad e^{in(x-y)}, \quad (25)$$

and is a function which is symmetric with respect to axes and diagonals.

Assume inductively that for all $k = 1, \dots, K-1$ the claim holds. In order to construct λ_K one needs that the monomials (25) do not show up in the right hand side of (24). Start with e^{inx} . Such a monomial can appear on the right hand side only from B_{pq} . But the functions λ_p, λ_q are even with respect to y , therefore B_{pq} is odd with respect to y and so the Fourier coefficient of e^{inx} must vanish. Analogously, e^{iny} can appear only from A_{pq} . But this is again an odd function on x and thus the Fourier coefficient of e^{iny} must vanish. In order to conclude about the monomials $e^{in(x+y)}, e^{in(x-y)}$ we notice that the equation (23) and therefore also (24) is invariant on the rotation of the plane by 45° and so the previous argument can be applied. Thus monomials (25) do not appear and λ_k can be found. One can easily see it is also symmetric with respect to the axes and diagonals. This proves the claim.

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